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Thermally sustained structure in convectively unstable systems

Robert J. Deissler
Institute for Computational Mechanics in Propulsion (ICOMP), Ohio Aerospace Institute, NASA Lewis Research Center, Cleveland, Ohio 44135

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The complex Ginzburg-Landau equation with a thermal noise term is studied under conditions when the system is convectively unstable. Under these conditions, the noise is selectively and spatially amplified giving rise to a noise-sustained structure. Analytical results, applicable to a wide range of physical systems, are derived for the variance, and the coefficients and thermal noise term are determined for Taylor-Couette flow with an axial through-flow. Comparison is made to recent experiments.

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Consider the equilibrium state of some spatially extended system and a small spatially localized perturbation about this state. If the perturbation grows at a fixed location in space, the equilibrium state is *absolutely unstable.* However, if the perturbation is convected with the mean flow such that it grows only in a moving frame of reference, eventually damping at any fixed location, the equilibrium state is convectively unstable [1-5]. Although little attention has been given to this distinction until fairly recently, it is an important distinction since these two types of instabilities give rise to qualitatively very different behavior. In a convectively unstable system external noise is selectively and spatially amplified giving rise to spatially growing waves and a *noise-sustained structure* [2-4], a concept introduced with studies of the complex Ginzburg-Landau equation. In contrast, in an absolutely unstable system structure is sustained by the internal dynamics.

Since noise—whether thermal or otherwise—is an element common to all physical systems and since any system with nonzero group velocity will be convectively unstable sufficiently close to and above onset on the instability [4,6], one would expect noise-sustained structure to be very common in nature. For example, in addition to being important in classic open-flow fluid systems such as jets, wakes, and channel flow [1-5,7], the above concepts are important in such diverse systems as film flow [8,9], binary fluid convection [6,10], sidebranching in dendrities [4,11,12], and traffic flow [4]. Considering the general nature of these concepts, there are undoubtedly systems in other fields to which they would also apply.

Since the complex Ginzburg-Landau (CGL) equation is a generic equation which describes systems near onset of an instability, it has proven to be an ideal system in which to explore these concepts. Also, since the equation is rather simple in form, there is some hope for deriving analytic results. Considering the fact that, up to this point, no analytic results have existed for convectively unstable systems in the presence of spatially extended noise, analytic results should prove very useful and would provide further insight into the interaction of noise with convectively unstable systems.

Recently, noise-sustained structure has been studied experimentally in Taylor-Couette flow with an imposed axial through-flow [13-15]. It was found that under convectively unstable conditions, noise-sustained structures of traveling vortices exist. This system is very useful for study in that, for sufficiently small Reynolds numbers, noise-sustained structure exists in a parameter regime where the flow is laminar and axisymmetric, thus allowing for a great deal of experimental control. Further, the CGL equation is valid for this system in a parameter regime of experimental interest.

Since in a convectively unstable system noise is amplified exponentially in space, even extremely low levels of noise will be sufficient to produce a noise-sustained structure, assuming the system is sufficiently long [2-4]. A very interesting question asked in Ref. [15] is whether or not the noise-sustained structure seen in the Taylor-Couette experiments is of thermal origin. This is an intriguing and important question since an affirmative answer would imply that further efforts at noise reduction—short of decreasing the temperature of the system—would have no effect on the flow. Also it could have important consequences for related systems such as Rayleigh-Bénard convection with through-flow [16,17]. Based on numerical solution of the CGL equation it was argued in Ref. [15] that thermal noise may play an important role in the Taylor-Couette experiments. However, as stressed by the authors, their result provides only an order-of-magnitude estimate since the noise term used in the CGL simulation was not derived rigorously from the Navier-Stokes equations. We note that thermal noise is also believed to be important in recent experiments in binary fluid convection [10].

In this paper we will first study, without reference to any particular physical systems, the CGL equation with a noise term that is δ-function-correlated in space and time. We will present analytic results for this system under conditions when the system is convectively unstable. Since the CGL equation is a generic equation, these results will be applicable to a wide variety of physical systems.

Next we will focus attention on the particular system mentioned previously (i.e., Taylor-Couette flow with an axial through-flow) and rigorously derive the noise term for the CGL equation, as well as the coefficients of the CGL equation (including the nonlinear term). Based on these results, it appears that the noise-sustained structure seen in the experiments is not of thermal origin. However, even if the structure in these experiments is not thermal in origin, it seems likely that experiments can be designed in which structure is sustained by thermal noise (i.e., molecular motion), considering the fact that the noise in these experiments is already so extremely small.

The complex Ginzburg-Landau equation [18-20] with a noise term is

$$A_t=\epsilon a A - v A_x + b A_{xx} - c|A|^2 A + \xi \;, \tag{1}$$

where a, b, and c are in general complex coefficients, v is the group velocity, $\epsilon \equiv R - R_c$ measures the "distance" above onset of the instability (where R and R_c are the control parameter and critical value of that parameter, respectively), $\xi(x,t)$ is a complex thermal noise term, and $A(x,t)$ is the slowly varying complex amplitude of a plane-wave solution at criticality. The conditions satisfied by ξ are $\langle \xi(x,t)\xi^*(x',t')\rangle = \sigma^2\delta(x-x')\delta(t-t')$, $\langle \xi(x,t)\xi(x',t')\rangle = 0$, and $\langle \xi(x,t)\rangle = 0$, where $*$ refers to the complex conjugate and δ is the Dirac delta function. These conditions will be satisfied if $\langle \xi_r(x,t)\xi_r(x',t')\rangle = \langle \xi_i(x,t)\xi_i(x',t')\rangle = (\sigma^2/2)\delta(x-x')\delta(t-t')$, $\langle \xi_r(x,t)\xi_i(x',t')\rangle = 0$, and $\langle \xi_r(x,t)\rangle = \langle \xi_i(x,t)\rangle = 0$, where the subscripts r and i refer to the real and imaginary parts, respectively. As shown by Graham [20], the fluid equations in the presence of thermal noise and near onset of the instability may be reduced to Eq. (1).

Assuming that $|A|$ is sufficiently small so that the nonlinear term may be neglected, the covariance function $K(x,x',t) \equiv \langle A(x,t)A^*(x',t)\rangle$ (where the angle brackets signify the expectation value) satisfies the following equation:

$$K_t = 2\epsilon a_r K - v(K_x + K_{x'}) + bK_{xx} + b^* K_{x'x'} + \sigma^2\delta(x-x') \;. \tag{2}$$

This equation was derived by discretizing the linear Eq. (1) in space, applying the results for a set of coupled ordinary differential equations [21], and then returning to the continuous spatial limit.

Assuming that the system is convectively unstable, i.e., $0 < \epsilon a_r < v^2 b_r/(4|b|^2)$ [2,3], the stationary solution of Eq. (2) for the semi-infinite interval $[0,\infty)$ with boundary conditions $K(x,0,t) = K(0,x',t) = 0$ is

$$K(x,x') = \frac{2\sigma^2\gamma}{\pi^2} e^{[vx/(2b) + vx'/(2b^*)]} \int_0^\infty dk \int_0^\infty dk' \frac{\sin(kx)\sin(k'x')}{\alpha^2 + bk^2 + b^* k'^2} \left[\frac{1}{\gamma^2 + (k-k')^2} - \frac{1}{\gamma^2 + (k+k')^2} \right] , \tag{3}$$

where $\gamma \equiv v b_r/|b|^2$ and $\alpha \equiv [v^2 b_r/(2|b|^2) - 2\epsilon a_r]^{1/2}$. After taking $x = x'$, making a change of variables, and performing one of the integrations, Eq. (3) may be reduced to

$$K(x,x) = \frac{\sigma^2}{\pi|b|}\gamma e^{\gamma x} \int_0^\infty ds \frac{\cos(sx) - \exp[-(|b|/br)(s^2+\eta^2)^{1/2}x]\cosh[(b_i/b_r)sx]}{(s^2+\eta^2)^{1/2}(s^2+\gamma^2)} , \tag{4}$$

where $\eta \equiv (\sqrt{2b_r}/|b|)\alpha$. This equation gives the variance $K(x,x) = \langle |A(x,t)|^2\rangle$ at the point x. For large x Eq. (4) may be written as the asymptotic series

$$K(x,x) \sim \frac{\sigma^2\gamma}{\sqrt{2\pi\eta}|b|} \left[\frac{1}{\gamma^2 - \eta^2} - \frac{1}{\gamma^2 + [(b_i/b_r)\eta]^2} \right] \frac{e^{(\gamma-\eta)x}}{\sqrt{x}} \sum_{l=0}^{n} \frac{\beta_l \Gamma(l+\frac{1}{2})}{\beta_0 \Gamma(\frac{1}{2}) x^l} , \tag{5}$$

where the β_l are the coefficients in the Taylor expansion

$$(1/\sqrt{2\eta+q})(\{2/[\gamma^2-(q+\eta)^2]\} - \{1/[\gamma^2+s_+(q)^2]\} - \{1/[\gamma^2+s_-(q)^2]\}) = \sum_{l=0}^{\infty} \beta_l q^l ,$$

where

$$s_\pm(q) = (b_i/b_r)(\eta+q) \pm (|b|/b_r)\sqrt{q(2\eta+q)} ,$$

where $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(l+\frac{1}{2}) = (l-\frac{1}{2})\Gamma(l-\frac{1}{2})$, Γ being the gamma function. The coefficient of x in the exponential, $\gamma - \eta$, is precisely twice the spatial growth rate of the amplitude of the most rapidly growing mode [2,3] as one would expect. Although this expression for the variance may look somewhat formidable, note that the leading term in the asymptotic series is given simply by replacing the sum in Eq. (5) by 1 and, as we shall see, this may be all that is necessary in many cases. Another feature that should be noted is that the variance does not increase purely exponentially in space for large x, but rather as $\exp[(\gamma-\eta)x]/\sqrt{x}$.

We now focus attention on Taylor-Couette flow with an axial through-flow. This system consists of two concentric cylinders with inner and outer radii r_i and r_o, respectively, with the inner cylinder rotating with velocity v_i, and with an imposed through-flow in the axial direction. The radial, azimuthal, and axial coordinates are denoted by (r,θ,x), respectively. In the parameter regime of interest, the flow will be axisymmetric and all derivatives with respect to θ will vanish. To derive the CGL equation the Navier-Stokes and continuity equations for the deviation of the velocity and pressure about the stationary background flow are written in cylindrical coordinates and the velocity, distance, time, and pressure are scaled with ν/d, d, d^2/ν, and ν^2/d^2, respectively, where d is the gap distance between the cylinders and ν is the kinematic viscosity. There are three independent parameters for this system: (1) the scaled inner cylinder velocity or azimuthal Reynolds number $\mathrm{Re}_{az} = v_i d/\nu$, (2) the scaled average axial velocity of the stationary background flow or axial Reynolds number $\mathrm{Re}_{ax} = \overline{W}d/\nu$, and (3) the scaled inner cylinder radius r_i/d, which may be written in terms of the radius ratio r_o/r_i as $r_i/d = 1/[(r_o/r_i) - 1]$.

As Re_{az} is gradually increased for a given Re_{ax} and r_i/d, at $\mathrm{Re}_{az,c}$, the critical value of Re_{az}, the stationary background flow becomes unstable. We define ϵ that appears in the CGL equation (1) as $\epsilon \equiv \mathrm{Re}_{az} - \mathrm{Re}_{az,c}$. This measures the "distance" above onset of the instability and is used as the expansion parameter in deriving the CGL equation. The linear coefficients a, v, and b are particularly easy to derive, being determined from the linear stability problem. Assuming a solution of the linearized fluid equations of the form $\exp[\lambda(k,\mathrm{Re}_{az})t+ikx]$, where t and x are the scaled time and axial position, respectively, these coefficients are given by $a=\partial\lambda/\partial\mathrm{Re}_{az}$, $v=-\partial\lambda_i/\partial k$, and $b=-\frac{1}{2}\partial^2\lambda/\partial k^2$, where the derivatives are evaluated at criticality (i.e., at k_c and $\mathrm{Re}_{az,c}$). The critical values of k and Re_{az} are determined from $\partial\lambda_r/\partial k=0$ and $\lambda_r=0$. The calculation of the coefficient of the nonlinear term, c, is much more involved and we only give a numerical value in this paper. For details of the derivation of coefficients of the CGL equation, the reader is referred to, for example, Refs. [18–20]. In terms of the amplitude A appearing in the CGL equation (1), the scaled radial, azimuthal, and axial fluid velocities and pressure are given by $(u,v,w,p)=A(x,t)(U,V,W,P)\exp(ik_cx-i\omega_ct)+\text{c.c.}$, where $U(r)$, $V(r)$, $W(r)$, and $P(r)$ are the radial eigenfunctions of the linear stability problem at criticality, k_c and ω_c are the critical wave number and frequency, respectively, and c.c. stands for the complex conjugate. Chebyshev polynomials were used in the radial direction giving highly accurate results.

In the presence of thermal noise, noise terms $N^{(i)}$ $(i=r,\theta,x)$ must be added to the Navier-Stokes equations for the radial, azimuthal, and axial velocity components, u, v, and w, respectively. From Landau and Lifshitz [22] we find that for thermal noise (writing the noise term in cylindrical coordinates)

$$N^{(i)}=(1/r)[(rS^{(ri)})_r+S^{(\theta i)}_\theta+(rS^{(xi)})_x+B^{(i)}] \quad (i=r,\theta,x), \tag{6}$$

where $B^{(i)}\equiv(-S^{(\theta\theta)},S^{(r\theta)},0)$, and where the correlations between the random components of the stress tensor (in scaled units) are

$$\langle S^{(ik)}(r,\theta,x,t)S^{(lm)}(r',\theta',x',t')\rangle =[2kT/(v^2\rho d)][\delta^{(il)}\delta^{(km)}+\delta^{(im)}\delta^{(kl)}+(\zeta-\tfrac{2}{3})\delta^{(ik)}\delta^{(lm)}]\delta(r-r')(1/r)\delta(\theta-\theta')\delta(x-x')\delta(t-t'), \tag{7}$$

where $\delta^{(ij)}$ is the Kronecker delta function $(i,j=r,\theta,x)$, k is Boltzmann's constant, T is the absolute temperature of the fluid, ρ is the fluid density, and ζ is the ratio of the second (or bulk) viscosity to the usual viscosity. Expanding the velocities and random components of the stress tensor in $e^{(im\theta)}$, where m corresponds to the azimuthal mode number, and noting that the $m=0$ mode dominates, we find $\delta(\theta-\theta')=1/(2\pi)$, in addition to all the derivatives with respect to θ vanishing.

To find the correlation of the noise term we followed Graham [20]. The correlation of the noise term is found to be

$$\langle\xi(x,t)\xi^*(x',t')\rangle=[Q/(2\pi|C|^2)]\delta(x-x')\delta(t-t')[(\tfrac{4}{3}+\zeta)I_1+(\tfrac{8}{3}-\zeta)I_2+(\zeta-\tfrac{2}{3})I_3+I_4], \tag{8}$$

where

$$I_1=\int dr(1/r)[|(1/r)U^\dagger-(dU^\dagger/dr)|^2+|(1/r)U^\dagger|^2],\quad I_2=\int dr(1/r)k_c^2|W^\dagger|^2,$$

$$I_3=\int dr(1/r)[|(1/r)U^\dagger-(dU^\dagger/dr)+ik_cW^\dagger|^2+|(1/r)U^\dagger-ik_cW^\dagger|^2-|(2/r)U^\dagger-(dU^\dagger/dr)|^2],$$

$$I_4=\int dr(1/r)[k_c^2|V^\dagger|^2+|(2/r)V^\dagger-(dV^\dagger/dr)|^2+|(1/r)W^\dagger-(dW^\dagger/dr)+ik_cU^\dagger|^2],\quad C=\int dr(U^\dagger U+V^\dagger V+W^\dagger W)$$

where $Q\equiv kT/(\rho dv^2)$ is a dimensionless number related to the thermal noise level and where $U^\dagger(r)$, $V^\dagger(r)$, and $W^\dagger(r)$ are the radial eigenfunctions of the adjoint stability problem at criticality.

Numerically evaluating the noise term for a radius ratio of $r_i/r_o=0.7376$ which corresponds to the experiments of Refs. [13, 15], and for an axial Reynolds number of $\mathrm{Re}_{ax}=3$, gives $\langle\xi\xi^{*\prime}\rangle=0.320\,23Q\delta(x-x')\delta(t-t')$. For these experiments we also have $d=0.6769$ cm, $\rho=1.04$ g/cm^3, $v=0.0158$ cm^2/s, and $T=293$ K which gives $Q=2.30\times10^{-10}$. The eigenfunctions were normalized such that $|U(r_h)|=1$, where $r_h=(r_i+r_o)/2$. Specifying how the eigenfunctions are normalized is essential when evaluating the noise correlation and the nonlinear coefficient, since the values of these quantities depend on the normalization of the eigenfunctions. The numerical coefficient of the noise correlation is rather insensitive to the value of Re_{ax} for small Re_{ax}, being, for example, 0.317 19 for $\mathrm{Re}_{ax}=0$. We note that the noise correlation is independent of ζ, which is probably a reflection of the incompressibility of the flow. This is fortunate since the second viscosity is unknown for liquids at low frequencies [23].

Evaluating the linear coefficients for $\mathrm{Re}_{ax}=3$ gives $a=0.314\,48+6.806\times10^{-3}i$, $v=3.690\,24$, and $b=1.910\,52+0.140\,70i$. The nonlinear coefficient is found to be $c=0.451\,27+4.444\times10^{-3}i$. We also find that $k_c=3.137\,13$, $\mathrm{Re}_{az,c}=84.3035$, and $\omega_c=11.0294$. The real parts of the coefficients a, b, and c, and k_c and $\mathrm{Re}_{az,c}$, are found to be rather insensitive to the value of Re_{ax} for small Re_{ax}, being, for example, $a_r=0.313\,72$, $b_r=1.910\,94$, $c_r=0.451\,92$, $k_c=3.136\,20$, and $\mathrm{Re}_{az,c}=84.0148$ for $\mathrm{Re}_{ax}=0$. The group velocity, the imaginary parts of the coefficients, and the critical frequency are 0 for $\mathrm{Re}_{ax}=0$ and are approximately proportional to Re_{ax} for small Re_{ax}.

Figure 1 (solid line) shows the root-mean-square (rms) average of the radial velocity at $r=r_h$ as a function of x from a numerical simulation of the CGL equation (1) using the above values for the coefficients and thermal noise term and using $\epsilon=4.668$. The rms average of the radial velocity at $r=r_h$ is related to the rms average of $|A|$ by $\langle u(r_h,x,t)^2\rangle^{1/2}=\sqrt{2}\langle|A(x,t)|^2\rangle^{1/2}$ by virtue of the normalization taken for the radial eigenfunctions [see discussion following Eq. (8)]. As can be seen, the noise is spatially amplified resulting in a noise-sustained structure.

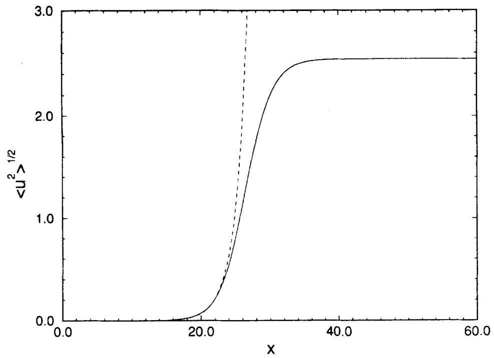

FIG. 1. rms average of the radial velocity at a radius midway between the cylinders from a numerical simulation of the CGL equation (solid line) and from the leading term (i.e., $n=0$) in the asymptotic series Eq. (5) (dashed line).

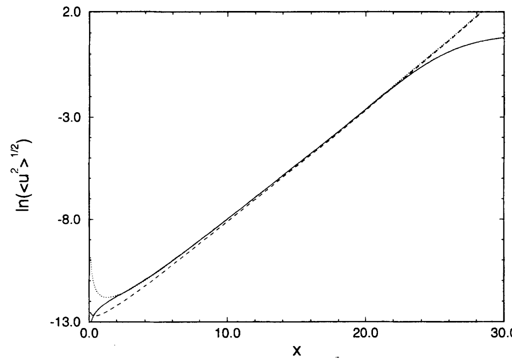

FIG. 2. Natural logarithm of the rms average of the radial velocity at a radius midway between the cylinders from a numerical simulation of the CGL equation (solid line), from the asymptotic series Eq. (5) for $n=0$ (dashed line), and from the asymptotic series Eq. (5) for $n=2$ (dotted line).

In the absence of a continuous source of noise, the solution is zero everywhere. Figure 1 (dashed line) shows $\langle u^2\rangle^{1/2}$ at $r=r_h$ as given by the leading term in Eq. (5). As can be seen, the agreement is excellent between the analytic result and the numerical solution of the CGL equation for small $|A|$. In order to see the range of x for which Eq. (5) is valid, Fig. 2 shows $\ln(\langle u^2\rangle^{1/2})$ from the numerical simulation (solid line), from the leading term in Eq. (5) (dashed line), and from Eq. (5) keeping three terms in the asymptotic series (i.e., $n=2$) (dotted line). Although the rms average as given by the leading term in Eq. (5) deviates somewhat from the numerical solution for smaller x, the rms average as given by the numerical solution and the rms average as given by Eq. (5) with $n=2$ show excellent agreement along most of the curve. Referring to Fig. 1 we see that the value of x at half maximum is about 26.5. For the experiments of Refs. [13, 15], this distance is about 21 [24]. Referring to Eq. (5) and noting that $(\gamma-\eta)/2=0.5625$, this implies that the noise amplitude in the experiments is about $(21/26.5)^{1/4}\exp[0.5625\times(26.5-21)]\approx 20$ times thermal noise.

In conclusion, we have studied the complex Ginzburg-Landau equation with a thermal noise term under conditions when the system is convectively unstable. Analytic results were derived for the variance. Since the CGL equation is a generic equation, and considering the facts that noise is an element common to all physical systems and all systems with nonzero group velocity are convectively unstable at onset of the instability, these results will be applicable to a wide variety of physical systems. The coefficients and thermal noise term for the CGL equation were determined for Taylor-Couette flow with an axial through-flow and comparison was made to experiment. Although the effective noise level in the experiments of Refs. [13,15] appears not to be thermal in origin, the noise level is nonetheless extremely small, being roughly an order of magnitude larger than thermal noise. Therefore, even if the structures in the experiments are not thermally sustained, it seems likely that an experiment can be designed in which thermally sustained structure does exist.

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